Last Time: Bases and Exchange.

Recall: If V is a vector space w/ finite basis B, then every basis of V has the same number of elements as B.

NB: We don't actually need the finiteness assumption... We won't (however) prove that :)

Def^n: Let V be a vector space. The *dimension* of V is the size of any of its bases.

Notation: $\dim(V)$ = dimension of V

Ex: Let $n \geq 0$. The dimension of $\mathbb{R}^n$ is n. because $\mathcal{E}_n = \{e_1, \cdots, e_n\}$ the standard basis, has n elts

Ex: Compute dimension of

$$V = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0 + a_1 = 0 = a_2 - a_3\} \leq \mathcal{P}_3(\mathbb{R}).$$

Sol: Let's compute a basis of V:

$$a_0 + a_1 = 0 \iff a_1 = -a_0$$

$$a_2 - a_3 = 0 \iff a_3 = a_2, \quad \text{So}$$

$$V = \{a_0 - a_0x + a_2x^2 + a_2x^3 : a_0, a_2 \in \mathbb{R}\}$$

$\therefore$ every polynomial in V has form:

$$a_0(1-x) + a_2(x^2+x^3).$$

Hence $B = \{1-x, x^2+x^3\}$ is a spanning set for V.

Check: B is lin ind.

Hence $B = \{1-x, x^2+x^3\}$ is a basis of V.

So $\dim(V) = \boxed{\# B} = |B| = 2$ number of elements in B. $\square$

Ex: Compute $\dim(V)$ for $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+b+c = 0 = a+b-c,\ d \in \mathbb{R} \right\}$

Sol: Compute a basis for V:

$$\left.\begin{array}{l} a+b+c=0 \iff a+b=-c \\ a+b-c=0 \iff a+b=c \end{array}\right\} \implies c=-c \iff c=0$$

$$\therefore\ V = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a+b=0 \cancel{= a+b},\ d \in \mathbb{R} \right\}$$

$$\therefore\ a+b=0 \iff b=-a$$

$$\therefore\ V = \left\{ \boxed{\begin{pmatrix} a & -a \\ 0 & d \end{pmatrix}} : a, d \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} : a, d \in \mathbb{R} \right\}$$

$B = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a spanning set for V.

B is also Lin. indep. Hence B is a basis, so $\dim(V) = \#B = 2$ $\boxtimes$

The following corollaries are nice exercises (all follow from the propositions proved last time).

Prop: Every vector space has a basis. ← Know this...

↳ Follows from Zorn's Lemma, which is equivalent to Axiom of Choice...) Don't need to know these...

Cor: Every independent set can be expanded to a basis.

Cor: Every spanning set can be reduced to a basis.

Cor: If $I \subseteq V$ is independent, then $\#I \le \dim(V)$

Cor: If V is finite dimensional, then every spanning set with $\dim(V)$ vectors is a basis.

Linear Maps

Recall: We've seen linear maps before: $\mathbb{R}^n \to \mathbb{R}^m$.
We'll extend the definition to arbitrary vector spaces:

Def^n: A function $L: V \to W$ of vector spaces is **linear** (i.e. a **linear map** or **linear homomorphism**) when for all $c \in \mathbb{R}$ and all $x, y \in V$ we have both:

$$L(cx) = cL(x) \quad \text{and} \quad L(x+y) = L(x) + L(y).$$

Ex: The projections are linear maps (i.e. maps which forget components).

$$\left.\begin{array}{l} p: \mathbb{R}^3 \to \mathbb{R}^2 \text{ w/ } p\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix} \\ q: \mathbb{R}^3 \to \mathbb{R}^2 \text{ w/ } q\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \\ s: \mathbb{R}^4 \to \mathbb{R} \text{ w/ } s\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = w \end{array}\right\} \text{ all linear!}$$

To see p is linear,

$$p\left(c\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = p\begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} = \begin{pmatrix} cx \\ cz \end{pmatrix} = c\begin{pmatrix} x \\ z \end{pmatrix} = c\,p\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$p\left(\begin{pmatrix} x_1 \\ y_1 \\ z \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z \end{pmatrix}\right) = p\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ z_2 \end{pmatrix}$$

$$= p\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + p\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$\therefore$ $p(cx) = cp(x)$ and $p(x+y) = p(x) + p(y)$ for all $c \in \mathbb{R}$ and $x, y \in \mathbb{R}^3$. Hence p is linear $\boxed{}$

Ex: The map $L : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3 : c + bx + ax^2 \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a linear map.

Earlier in the course, we proved the following:

*Lem: If $L : V \to W$ is linear, then $L(0_V) = 0_W$.

Prop (Alt. Characterization of Linear Maps): Let $L : V \to W$ be a function. The following are equivalent:

1. L is a linear map.
2. For all $c \in \mathbb{R}$ and all $x, y \in V$, we have both $L(cx) = cL(x)$ and $L(x+y) = L(x) + L(y)$.
3. For all $c \in \mathbb{R}$ and all $x, y \in V$, we have $L(x + cy) = L(x) + cL(y)$. ← easiest condition to check...
4. For all $(n \geq 2)$ $c_1, c_2, \cdots, c_n \in \mathbb{R}$ and all $x_1, x_2, \cdots, x_n \in V$ we have

useful theoretically.

$$L(c_1 x_1 + c_2 x_2 + \cdots + c_n x_n) = c_1 L(x_1) + c_2 L(x_2) + \cdots + c_n L(x_n).$$

(i.e. L preserves all linear combinations).

Exercise: Rework the old proofs into proofs for this case...

Ex: Is $L : \mathcal{P}_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ w/

$$L(c + bx + ax^2) = \begin{pmatrix} a & b \\ c & a+b \end{pmatrix}$$ linear?

Sol: We check our condition:

$$L\left((c_1 + b_1 x + a_1 x^2) + d(c_2 + b_2 x + a_2 x^2)\right) \stackrel{?}{=} L(c_1 + b_1 x + a_1 x^2) + d L(c_2 + b_2 x + a_2 x^2)$$

$$L\left((c_1 + b_1 x + a_1 x^2) + d(c_2 + b_2 x + a_2 x^2)\right)$$

$$= L\left((c_1 + dc_2) + (b_1 + db_2)x + (a_1 + da_2)x^2\right)$$

$$= \begin{pmatrix} a_1 + da_2 & b_1 + db_2 \\ c_1 + dc_2 & (a_1 + da_2) + (b_1 + db_2) \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & b_1 \\ c_1 & a_1 + b_1 \end{pmatrix} + \begin{pmatrix} da_2 & db_2 \\ dc_2 & da_2 + db_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & b_1 \\ c_1 & a_1 + b_1 \end{pmatrix} + d\begin{pmatrix} a_2 & b_2 \\ c_2 & a_2 + b_2 \end{pmatrix}$$

$L(c + bx + ax^2) = \begin{pmatrix} a & b \\ c & a+b \end{pmatrix}$

$$= L(c_1 + b_1 x + a_1 x^2) + d\,L(c_2 + b_2 x + a_2 x^2)$$

Hence L is linear by Alt. Char. of Linearity. $\boxed{//}$

Non-Ex: The map $L : \mathbb{R}^3 \to \mathbb{R}^2$ w/ $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ z + 1 \end{pmatrix}$ is NOT linear!

Lem: $L(0_V) = 0_W$ for linear $L : V \to W$.

$$L\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ 0 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

So $L(0_{\mathbb{R}^3}) \neq 0_{\mathbb{R}^2}$ yields L is not linear.

Prop: Let $L : V \to W$ be linear and let V have a basis $\mathcal{B}$. Then L is determined by its action on $\mathcal{B}$.

Point: Given $v \in V$, $v = \sum_{i=1}^{n} c_i b_i$. Thus:

$$L(v) = L\left(\sum_{i=1}^{n} c_i b_i\right)$$

$$= L(c_1 b_1 + c_2 b_2 + \cdots + c_n b_n)$$

$$= c_1 L(b_1) + c_2 L(b_2) + \cdots + c_n L(b_n).$$

__Prop:__ Let V, W be vector spaces. Let $\mathcal{B}$ be a basis of V. Every function $f : \mathcal{B} \to W$ extends (linearly) to a linear map $F : V \to W$. Indeed:

$$F\left(\sum_{i=1}^{n} c_i b_i\right) = \sum_{i=1}^{n} c_i f(b_i).$$

__Point:__ Given a function associating vectors of basis $\mathcal{B}$ to vectors of W, there is a corresponding induced linear map...

__Ex:__ Let $V = \mathbb{R}^3$ and $W = \mathcal{M}_{2\times 3}(\mathbb{R})$.

Define $f : \mathcal{E}_3 \to W$ by:

$$f(e_1) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$f(e_3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$ The induced map

$F : \mathbb{R}^3 \to \mathcal{M}_{2\times 3}(\mathbb{R})$ is

$$\begin{aligned} F\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= F(xe_1 + ye_2 + ze_3) \\ &= xf(e_1) + yf(e_2) + zf(e_3) \\ &= x\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} + y\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + z\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x+z & 0 & 2x+y+z \\ 0 & x+z & x+y \end{pmatrix} \end{aligned}$$

And F is a linear map! $\square$